APPENDIX

VOICE-LEADING SUMS UNDER TRANSLATION AND TYMOCZKIAN PERMUTATION OF CHORDS

1. Klangs and their Cyclic Permutations

DEFINITION 1.1. Fix $a > n \geq 2$, $a, n \in \mathbb{Z}$.
We shall refer to an ordered set K of n integer classes (mod a), as *a klang*.

DEFINITION 1.2. Let $K = (\kappa_0, \kappa_1, \dots, \kappa_{n-1})$, $K' = (\kappa'_0, \kappa'_1, \dots, \kappa'_{n-1})$, be klangs.
We shall write $K' = \mathrm{P}(K)$ if $\kappa'_i = \kappa_{i+1 \text{ (mod } n)}$.

Note:

- P is a bijection on the set of klangs, hence the inverse function P^{-1} is defined.
- The group $(\mathrm{P}^q, \circ)$ of operations on $\{K\}$, $\mathrm{P}^q = \overbrace{\mathrm{P} \circ \dots \circ \mathrm{P}}^{q \text{ times}}$ if $q \geq 1$, $\mathrm{P}^q = \overbrace{\mathrm{P}^{-1} \circ \dots \circ \mathrm{P}^{-1}}^{|q| \text{ times}}$ if $q \leq -1$, is a group of cyclic permutations.

2. Chords, their Translations and "Tymoczkian Permutations"

DEFINITION 2.1. Let $K = (\kappa_i)$ be a klang.
We shall refer to an ordered set $C = (c_i)$ of n integers as a *chord representing* K, if $c_i \equiv \kappa_i$ (mod a).

DEFINITION 2.2. Let $C = (c_0, c_1, \dots, c_{n-1})$, $C' = (c'_0, c'_1, \dots, c'_{n-1})$, be chords representing the klangs K, K', respectively. We shall write:

(A) $C' = \mathbb{T}(C)$ if $c'_i = c_i + 1$.
(B) $C' = \mathbb{P}(C)$ if both
  (1) $K' = \mathrm{P}(K)$ and
  (2) $c_i < c'_i < c_i + a$.

Note:

- $\mathbb{T}$ and $\mathbb{P}$ are bijections on the set of chords, hence the inverse functions $\mathbb{T}^{-1}$ and $\mathbb{P}^{-1}$ are defined.
- $\mathbb{T}^t\big(\mathbb{P}^p(C)\big) = \mathbb{P}^p\big(\mathbb{T}^t(C)\big) = \mathbb{T}^t \circ \mathbb{P}^p(C)$, where $\mathbb{T}^t = \overbrace{\mathbb{T} \circ \ldots \circ \mathbb{T}}^{t \text{ times}}$ if $t \geq 1$, $\mathbb{T}^t = \overbrace{\mathbb{T}^{-1} \circ \ldots \circ \mathbb{T}^{-1}}^{|t| \text{ times}}$ if $t \leq -1$, and similarly for $\mathbb{P}^p$.
- The group $(\mathbb{T}^t,\circ)$ of operations on $\{C\}$ is a group of translations; the group $(\mathbb{P}^p,\circ)$ is termed in this review "Tymoczkian permutations."*

3. Standard Klangs and Voice-Leading Sums

DEFINITION 3.1. We shall refer to a klang $K = (\kappa_i)$ as *standard* if there exist representatives $\bar{\kappa}_i$ of κ_i such that

$$\bar{\kappa}_0 < \bar{\kappa}_1 < \cdots < \bar{\kappa}_{n-1} < \bar{\kappa}_0 + a.$$

DEFINITION 3.2. Let $C = (c_0, c_1, \ldots, c_{n-1})$ and $C' = (c'_0, c'_1, \ldots, c'_{n-1})$ be chords. We shall write

$$\sum_{i=0}^{n-1} (c'_i - c_i) = VLS$$

and shall refer to VLS as the voice-leading sum from C to C'.

THEOREM 3.1. Let K be a klang.
Then the following are equivalent:

(1) K is standard.
(2) For any chord C representing K, and for any p and t, letting $C' = \mathbb{P}^p \circ \mathbb{T}^t(C)$, we have:

$$VLS = ap + nt.$$

* $(\mathbb{P}^p,\circ)$ is not truly a group of permutations, hence "Tymoczkian permutations" is, strictly speaking, a misnomer. But since these operations are partially determined by the cyclic permutations P^q (see Definition 2.2(B)), the terminological transgression, I hope, may be excused.

Proof. Since the effect of arbitrary t on *VLS* is obvious, we shall only consider the effect of p. Moreover, it suffices to treat $p = 1$, since the rest will follow by induction. That is, we only need to prove that if $C' = \mathbb{P}^1(C)$, then K is standard if, and only if, $VLS = a$.

Denote $\bar{c}_i$ the representative of c_i in the interval $[0, a)$, that is, $c_i = \bar{c}_i + m_i a$ for some integer m_i, $0 \leq \bar{c}_i \leq a - 1$. Using Definition 2.2(B) we have:

$$c_i' = \begin{cases} \bar{c}_{i+1} + m_i a \text{ if } \bar{c}_i < \bar{c}_{i+1}, \\ \bar{c}_{i+1} + (m_i + 1)a \text{ if } \bar{c}_{i+1} < \bar{c}_i. \end{cases}$$

The number of indices i (mod n) such that $\bar{c}_{i+1} < \bar{c}_i$ is at least 1, and is exactly 1 if and only if there exists an index j such that $\bar{c}_j < \bar{c}_{j+1} < \cdots < \bar{c}_{n-1} < \bar{c}_0 < \cdots < \bar{c}_{j-1}$. This last condition is easily seen to be equivalent to the condition that K is standard. This proves the theorem.